Open-Ended Problem Solving in a High School Calculus Class

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Introduction:

In my last four years of teaching, I have noticed that students consistently struggle with solving problems that unfamiliar to them, despite having all of the tools necessary to do so. This was especially true for theoretical mathematical problems. Watching Dan Meyer's TED talk was eye-opening for me, because the five symptoms of poor reasoning described my students. (Meyer, 2010) By listening carefully to the questions they asked me and each other, I learned that for many of my students, math was a collection of seemingly random rules. They knew which rules to follow based on a mental catalog of problems they had seen before. When I asked them to extend what they had learned to problems or contexts they hadn't seen before, they had no idea which rules to apply, or how to figure that out for themselves. Based on Meyer's suggestions, I decided to try using open-ended and real-world problems to help my students improve their reasoning and problem-solving skills like pulling together disparate information, trying different approaches, patience with irresolution, inductive reasoning and mathematical communication.

Math Question:

How well does using open-ended and real-world problems help senior calculus students develop problem solving skills?

Literature Review:

There is reform happening in math classrooms across the United States, from the elementary level to the university level. The problem being addressed is the one discussed above – our students are not learning to reason mathematically or problem solve effectively, they're merely learning procedures. (Meyer, 2010) In response to this, educators are incorporating more and more open-ended and real-world problems into their curriculums. Calculus courses at Duke University were taught using real data and computer modeling programs. Students work in pairs to solve problems using these tools, and then write lab reports in proper English. (Culotta, 1992) Schwalbach and Dosemagen (2000) rearranged a calculus course to match the physics curriculum at a Midwest school to provide concrete context for calculus concepts in order to help her students develop deep understanding. Meyer (2011a, 2011b) advocated showing students a visual to provoke meaningful and perplexing questions in an effort to involve students in problem formulation, then scaffolding students through the rest of the problem-solving process.

Students were evaluated using formative and performance assessments, and self-reflections on their learning. They were expected to be able to extend their understanding to new problems, as well as write clear explanations of their process and results. (Culotta, 1992; Schwalbach & Dosemagen, 2000) Though individual rubrics differ, there is some consistency in the specific areas students are evaluated on. For example: clear statement/understanding of the problem to be solved, using problem solving strategies, properly using mathematical techniques, communication of reasoning, and evaluation of the results. (Association of American Colleges and Universities [AAC&U], 2009; Chicago Public Schools [CPS], 1994; Foster, 2000)

The literature indicates that such an approach to math curriculum does improve student understanding and ability, though the results are anecdotal or not generalizable. It is notable that while students' ability to problem solve and understand appeared to increase, the students themselves were unhappy with the changes to the curriculum at first. They did, however, eventually come to appreciate it. (Culotta, 1992)

Data Collection:

My calculus class consists of 11 senior girls, some of whom have been in the honors track for math their entire math career, and others who have been in the regular track and doubled up on math courses in order to take calculus as a senior. Two of the girls had needed a remedial math course before they entered freshman algebra. Another two students have 504 accommodation plans for language processing difficulties, both written and auditory. The course itself is taught in a block period of 80 minutes over 1 semester.

I started assigning open-ended problems to my students at the beginning of the year. On August 31, I introduced my students to their Research Logs by discussing the guidelines for working in them, looking through sample work (from my college notebook) and comparing it to the guidelines, and then a problem for them to try. For this first problem, I used Dan Meyer's "An Obscure Geometry Problem" video, where students are shown a square divided into four equal parts by the two diagonals, the end of one of the diagonals moving down from the corner to the midpoint of the side, and then asked to find the percentages of the square that the new areas filled. Students took the problem home, worked on it individually, and brought their work to class to share in groups of 3 or 4 students. As a group, they were to ask questions about each student's work, critique it, share ideas of what else to try, and then work together on one of the ideas. During this time, I moved around the room, noting what was working well in the small groups and what was not. Then we gathered as a whole group, where I shared with them what I had noted and offered some suggestions. Once we had discussed how they worked in small groups, I asked each group to share one idea from their discussion. The other groups were again to ask questions and offer constructive criticism or suggestions to further the work. Students then took the problem home as homework for a second evening, armed with ideas from the discussions. These procedures were repeated the next day, so that possible solutions could again be evaluated by first the small groups, then the whole class.

On September 16, I assigned the second open-ended problem, where I asked them how to find the slope of a curve at any of three specified points on it. We proceeded in the same manner we had previously: working individually at home, discussing in groups, then sharing with the entire class. This time, however, students had a harder time generating ideas to pursue the first night, and so came to class with little to discuss in the small groups. Instead, the small groups were where the ideas were really generated, and the whole class sharing made sure everyone had something to work with the second night. On the second day, students had work that could be questioned and critiqued. So I asked them to spend one more night working toward a solution. The third day, students shared again, but had not really refined their ideas much beyond what they had been the previous day. I encouraged them to use their intuitive (steepness) and algebraic (linear) understanding of slope to compare their work to the actual curve. A half-hearted attempt was made, but then my students declared they were stuck. Since it was limit reasoning that they were missing, I suggested we let the problem rest, and learn some new tools that might help.

We returned to the Slope of a Curve problem on November 8, after completing a unit on limits. Students were vocal about disliking the problem, but began to work on it again. Again, we followed the same pattern of individual work, group discussion and class sharing for two days. It was clear the first day that several still lacked understanding of the more conceptual topics like the Intermediate Value Theorem and continuity, because they indiscriminately tried to apply these ideas, without stating why or how they thought the ideas could be useful. Others immediately jumped into using limits, but took the limit of the function without first considering that it would only give them the y-coordinates of the points x was approaching. With these misunderstandings and their continued frustration in mind, I created a page of several questions to help them pinpoint why the slope of a curve at a singular point was such a problem. (Namely, you get $\frac{0}{0}$ when you use the algebraic slope formula with one point.) The next day, I gave them the questions to discuss with their small groups. One group finally suggested that the limit was a lot like their idea of looking at the slopes of secant lines that went through points on either side of the specified point. This led another group to ask if it was possible to take the limit of the slope formula when it had two x's and y's in it. I made this the new focus question for all group discussions, and a third group determined it would work if you made one of the points in the slope equation the specified points. When I asked students to critique this possible solution, they could offer no arguments against it, but were not excited that they had (as a class) solved such a difficult problem. They said they were just glad to be done with it.

The final problem was created by them on November 17. The night before, for homework, they were asked to watch a short video of a car crash, and develop a list of questions. In class, I led a discussion about which questions they thought we could answer by analyzing the video. Once we had generated a short list, I gave them their new small groups and asked each group to decide on a question from that list to answer and then work on answering it. They took the problem home to work on individually, then returned the second day to share with their group and discuss further.

To gather numerical data on students' written work, I developed a Problem Solving Rubric based on rubrics created by the Association of American Colleges and Universities and Thomas Foster, and rubrics collected by Chicago Public Schools. Students were scored out of 20 points, based on four categories: problem solving strategies, communication, evaluation of results, and competent use of mathematics. The first three categories assign one point per criteria the student met. The fourth category (competent use of mathematics) is a ranking scale, based on the lowest criteria observed. If a student had correct mathematics in one section of her work but still used "math magic" in another, she was ranked as level 2. The rubric is as follows:

Problem Solving Rubric:

Problem solving strategies

- ☐ The student indicates that she made a plan before looking at examples.
- ☐ The student continued to create and look at examples until she noticed something.
- ☐ The student examined the examples in sufficient detail.
- ☐ The student created and examined the examples with care.
- ☐ The student used some sort of representation(s) in her work (graphs, diagrams, pictures, and equations).

Communication

- ☐ The student wrote a few *sentences* explaining what she was about to do.
- ☐ The student wrote *sentences* describing what she noticed about the examples.

-OR-

If the student was confused, did she note at which point she was confused and write some *questions* (for her group or for the class) that she would like answered.

- The student clearly labels her representations and defines any variables.
- There is organization to the student's work; one step flows into the next.
- The student provided supporting arguments or reasoning for what she observed.

Evaluation of results

- The student considers how to and tries to refine her observations.
- The student formulated her observations into a general rule.
- The student attempted to find counterexamples to her rule.
- The student explains why her rule is correct.
- The student considers possibilities for extensions and further work.

Competent use of Mathematics Techniques

(1) Solution terminates for no apparent reason

(2) When an obstacle happens, "math magic" or other unjustified relationships occurs

(3) When an obstacle happens, solution stops.

(4) Math errors (sign errors, distribution errors, etc.)

(5) Mathematics is correct.

I also asked students to write a self-reflection of their learning and work with the research log. I used the questions listed below as prompts, as well as inviting them to share any other constructive thoughts on problem solving or the Research Log.

- Do you feel that you have improved since the beginning of the year?
- Are you more comfortable approaching problems you know little about initially?
- What strategies have you used?

- Have you learned any new strategies?
- Do you think working on Research Log problems has been helpful?
- Are there skills listed on the Problem Solving Rubric you still feel you need to work on?
- Are there skills you feel are needed that aren't on the rubric?

Data Analysis

After all of the problems had been completed, I used the Problem Solving Rubric to score the students' work on all 4 problems. The scores (out of 20) are listed in Table 1.

Table 1: Students' Problem Solving Scores

Student	Problem 1	Problem 2	Problem 2 (revisited)	Problem 3
A	8	3	3	1
B	7	12	3	3
C	11	11	9	6 (no math)
D	11	12	10	5 (no math)
E	11	11	8	10
F	4 (no math)	6	2	2 (no math)
G	11	12	1	2 (no math)
H	7	4	1 (no math)	2 (no math)
I	15	10	9	4 (no math)
J	12	13	11	4 (no math)
K	9	11	12	4

("no math" indicates that the student did not include any calculations, and could therefore not be scored in the fourth category.)

It appears from the numerical data that the students' problem solving skills decreased over time, which is counter to what the reviewed literature indicated. However, as the semester progressed, students stopped fully writing out their thoughts and reasoning as they worked, so their later written work may not reflect their processes as well as their early work. If we consider this in terms of the students' responses on the surveys, it was not a decrease in their general skills that led to the decreasing scores. The most common sentiment on the surveys was that they were frustrated. For example, Student A wrote, "I just feel frustrated to start out with which doesn't make me want to actually try the problem." Student E shared:

> "I still get really frustrated when I spend a lot of time on something and I never seem to get it and I cannot seem to find any way to do it… I do not think it has been helpful because all it has done is make me stressed, discouraged, and angry. I get discouraged because I get so frustrated to the point where I just give up because I cannot seem to get any result…"

Students stopped putting in the effort to write down all of their reasoning because they were frustrated with a lack of early success on the problems. Coupled with a lack of solution when we moved on from the first attempt at Slope of a Curve problem, it is easy to see where students who are impatient with irresolution would be frustrated and want to give up.

Several students did find that working in the research log had improved their willingness to attempt an unfamiliar problem. Student K reported, "I think my comfort level has increased with problems I have never seen before," and Student F stated, "When it comes to math, I still have problems approaching foreign problems but I am becoming more open to problems." For these students, the time it would take for them to give up was extended, but not beyond the time it would take for the other frustrated students to give up.

As was the case with Duke's introduction of a problem-based calculus course, my students were uncomfortable with the non-traditional aspects of these open-ended problems. They asked several times if (or when) I was going to tell them the answer. Student J indicated on her survey that she felt she could learn "in a more effective and in a less time-consuming way" than the Research Log. Student H wrote, "I do not learn from exploring on my own, I learn best from people who already know how to do it. In addition, the research logs are not a good use of time I'd rather be learning from a teacher and not doing it on my own."

All in all, it appears that these problems, approached in this way, did not help my students improve their problem-solving skills.

Future Actions:

The results of this research project were disappointing. However, that does not mean that using open-ended problems won't help students develop reasoning and problem-solving skills. I need to reassess my approach, the amount of scaffolding and support needed by students to be successful and the way the problems are cast, and restructure the Research Log procedures to match.

Dan Meyer's advice to teachers who lead this type of learning is "Be less helpful." There is a fine balance between being less helpful and supporting students so they will feel successful. I apparently didn't find the proper balance for the students I have this year. I think I need to wean students away from relying on a teacher. The advice should be "Be less helpful as the course continues." But, the helpfulness should be related to problem-solving in general, not the problems specifically. For example, it would be wise to suggest different problem solving strategies to help students who feel immediately helpless find something to try. Or instruct students on how to evaluate their results by modeling it first, then making suggestions for how to do so when they ask, asking leading questions, or asking other students how they would recommend evaluating the results.

In order for students to be motivated in their work, they need to achieve success early enough that they feel they can solve the problems. At the same time, the success cannot come so easily that they feel there is no point to the problem they are trying to solve. To facilitate early experiences with success, the early problems I pose in the future will need to be very accessible to students. The first question was actually a good question, because it was easy to understand, and all of the students had had a geometry course and a trigonometry course. They had obvious tools to draw on (as well as some that were not so obvious to them – like the Cartesian plane and the distance formula). A similarly approachable problem is the famous Bridges of Konigsberg problem. Students can jump right in by drawing different routes to get a feeling if it was possible to all of the bridges once, or not. Once they're convinced it isn't possible, they begin looking for the reason why all on their own. Next year, I will be including that problem, as well as several others that may have little to do with calculus, but will help my students understand what it is to work through an open ended problem.

Another way to motivate students is in how the question is presented to them. I will need consider how I can recast the questions I asked my students to solve so that they will want to solve them, not just do so because it is a requirement of the course.

I also need to determine if the individual, group and class pattern could be better ordered. One possibility is to start with the whole class brainstorming about how to approach the problem, then grouping them by the approach they prefer. Another possibility is assigning students to study groups that are required to meet outside of class to work on the research log problems. Both of these techniques allow students someone to bounce ideas off of before they do any independent work. The hope is that it would help combat the feeling of helplessness some students feel when they first approach unfamiliar problems.

Finally, I need to re-evaluate the Problem Solving Rubric itself. Is the mathematical computation section a necessary component? It doesn’t allow for students to reason mathematically without doing any computation. Should some of the skills be more heavily weighted? Is the communication section as important as the strategies and evaluation sections? Finally, the rubric seems better suited to theoretical problems than more concrete problems. Not all problems will require students to generate a general rule for the situation. They may be trying to find concrete values. The language of the rubric needs to be changed to allow for that possibility.

In the end, I think this was a vital process to go through. I started working on open-ended problem solving with my students before learning about the action research process. If I hadn’t had to analyze my students work in this way, I would not know the extent of the changes I need to consider making.

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